



11.5 Self-adjoint and skew-adjoint linear operators





Introduction

- In this topic, we will
 - Define self-adjoint and skew-adjoint linear operators
 - See that symmetric real matrices and conjugate symmetric complex matrices are self-adjoint
 - See that skew-symmetric real matrices and conjugate skew-symmetric complex matrices are skew-adjoint
 - Learn that every linear operator is the sum of a self-adjoint and a skew-adjoint linear operator





Definitions

- A linear operator $A: \mathcal{U} \rightarrow \mathcal{U}$ is said to be *self-adjoint* if

$$\langle A\mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{u}, A\mathbf{v} \rangle$$

for all vectors $\mathbf{u}, \mathbf{v} \in \mathcal{U}$

- A linear operator $A: \mathcal{U} \rightarrow \mathcal{U}$ is said to be *skew-adjoint* if

$$\langle A\mathbf{u}, \mathbf{v} \rangle = -\langle \mathbf{u}, A\mathbf{v} \rangle$$

for all vectors $\mathbf{u}, \mathbf{v} \in \mathcal{U}$





Applications

- Self-adjoint operators are critical in your course in quantum mechanics
 - Operators that extract information about position, momentum, energy, etc. of particles must be self-adjoint





Real self-adjoint operators

- The adjoint of a real matrix is the transpose of that matrix
 - Given a square matrix A , then the entries of the transpose are:

$$\left(A^T\right)_{i,j} = a_{j,i}$$

- For these to be equal, we require

$$a_{i,j} = a_{j,i}$$

- In other words, a real square matrix is self-adjoint if the entry above the diagonal equals the entry in its reflection through the diagonal
 - Because of this property, we call such matrices *symmetric*





An example of a symmetric matrix

- The following matrix is symmetric,
and is therefore self-adjoint:

$$A = \begin{pmatrix} 3.2 & -0.3 & 0.7 & 0.2 \\ -0.3 & 4.5 & -1.8 & 0.9 \\ 0.7 & -1.8 & 2.4 & 6.8 \\ 0.2 & 0.9 & 6.8 & -0.4 \end{pmatrix}$$





Example

- Let's check if that's actually true:

$$A = A^T = \begin{pmatrix} 3 & 1 & -1 & -2 \\ 1 & 2 & 2 & 1 \\ -1 & 2 & -5 & -3 \\ -2 & 1 & -3 & 4 \end{pmatrix} \quad \mathbf{u} = \begin{pmatrix} -1 \\ 3 \\ 1 \\ -2 \end{pmatrix} \quad \mathbf{v} = \begin{pmatrix} 2 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

$$\langle A\mathbf{u}, \mathbf{v} \rangle = \left\langle \begin{pmatrix} 3 \\ 5 \\ 8 \\ -6 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right\rangle = -8$$

$$\langle \mathbf{u}, A\mathbf{v} \rangle = \left\langle \begin{pmatrix} -1 \\ 3 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 5 \\ 1 \\ 0 \\ 3 \end{pmatrix} \right\rangle = -8$$





Example

- If we change even one value, it no longer holds:

$$A^T \neq A = \begin{pmatrix} 3 & 1 & -1 & -2 \\ 1 & 2 & 2 & 1 \\ -1 & 2 & -5 & -3 \\ -2 & 1 & -2 & 4 \end{pmatrix} \quad \mathbf{u} = \begin{pmatrix} -1 \\ 3 \\ 1 \\ -2 \end{pmatrix} \quad \mathbf{v} = \begin{pmatrix} 2 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

$$\langle A\mathbf{u}, \mathbf{v} \rangle = \left\langle \begin{pmatrix} 3 \\ 5 \\ 8 \\ -5 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right\rangle = -7 \neq \langle \mathbf{u}, A\mathbf{v} \rangle = \left\langle \begin{pmatrix} -1 \\ 3 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 5 \\ 1 \\ 0 \\ 2 \end{pmatrix} \right\rangle = -6$$





Example

- If, however, we use the transpose, we get the same result

$$A^T \neq A = \begin{pmatrix} 3 & 1 & -1 & -2 \\ 1 & 2 & 2 & 1 \\ -1 & 2 & -5 & -3 \\ -2 & 1 & -2 & 4 \end{pmatrix} \quad \mathbf{u} = \begin{pmatrix} -1 \\ 3 \\ 1 \\ -2 \end{pmatrix} \quad \mathbf{v} = \begin{pmatrix} 2 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

$$\langle A\mathbf{u}, \mathbf{v} \rangle = \left\langle \begin{pmatrix} 3 \\ 5 \\ 8 \\ -5 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right\rangle = -7 = \langle \mathbf{u}, A^T \mathbf{v} \rangle = \left\langle \begin{pmatrix} -1 \\ 3 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 5 \\ 1 \\ 1 \\ 3 \end{pmatrix} \right\rangle = -7$$





Complex self-adjoint operators

- The adjoint of a complex matrix is the conjugate transpose of that matrix
 - Given a square matrix A , then the entries of the conjugate transpose are:
$$\left(A^*\right)_{i,j} = a_{j,i}^*$$
 - For these to be equal, we require
$$a_{i,j} = a_{j,i}^*$$
 - In other words, a complex square matrix is self-adjoint if the entry on or above the diagonal equals the conjugate of the entry in its reflection through the diagonal
 - Because of this property, we call such matrices *conjugate symmetric*





Complex self-adjoint operators

Theorem

The diagonal entries of a conjugate symmetric matrix are real.

Proof:

If $a_{i,j} = a_{j,i}^*$, then if $i = j$, it follows that

$$a_{i,i} = a_{i,i}^*$$

This is only true if the diagonal entries are all real. ■





Example

- Let's check if that's actually true:

$$A = A^* = \begin{pmatrix} -1 & 1-j & 2+j \\ 1+j & 2 & 1 \\ 2-j & 1 & -2 \end{pmatrix} \quad \mathbf{u} = \begin{pmatrix} 1 \\ 1-j \\ 2 \end{pmatrix} \quad \mathbf{v} = \begin{pmatrix} 2+j \\ 1 \\ j \end{pmatrix}$$

$$\langle A\mathbf{u}, \mathbf{v} \rangle = \left\langle \begin{pmatrix} 3 \\ 5-j \\ -1-2j \end{pmatrix}, \begin{pmatrix} 2+j \\ 1 \\ j \end{pmatrix} \right\rangle = 9+3j = \langle \mathbf{u}, A\mathbf{v} \rangle = \left\langle \begin{pmatrix} 1 \\ 1-j \\ 2 \end{pmatrix}, \begin{pmatrix} -2 \\ 3+4j \\ 6-2j \end{pmatrix} \right\rangle$$





Example

- If the matrix is symmetric and not conjugate symmetric, it no longer holds:

$$A^* \neq A = \begin{pmatrix} -1 & 1-j & 2+j \\ 1-j & 2 & 1 \\ 2+j & 1 & -2 \end{pmatrix} \quad \mathbf{u} = \begin{pmatrix} 1 \\ 1-j \\ 2 \end{pmatrix} \quad \mathbf{v} = \begin{pmatrix} 2+j \\ 1 \\ j \end{pmatrix}$$

$$\langle A\mathbf{u}, \mathbf{v} \rangle = \left\langle \begin{pmatrix} 3 \\ 5-3j \\ -1 \end{pmatrix}, \begin{pmatrix} 2+j \\ 1 \\ j \end{pmatrix} \right\rangle = 11+5j$$

$$\langle \mathbf{u}, A\mathbf{v} \rangle = \left\langle \begin{pmatrix} 1 \\ 1-j \\ 2 \end{pmatrix}, \begin{pmatrix} -2 \\ 5 \\ 4+2j \end{pmatrix} \right\rangle = 11+9j$$





Example

- If, however, we use the conjugate transpose, the values are equal

$$A^* \neq A = \begin{pmatrix} -1 & 1-j & 2+j \\ 1-j & 2 & 1 \\ 2+j & 1 & -2 \end{pmatrix} \quad \mathbf{u} = \begin{pmatrix} 1 \\ 1-j \\ 2 \end{pmatrix} \quad \mathbf{v} = \begin{pmatrix} 2+j \\ 1 \\ j \end{pmatrix}$$

$$\langle A\mathbf{u}, \mathbf{v} \rangle = \left\langle \begin{pmatrix} 3 \\ 5-3j \\ -1 \end{pmatrix}, \begin{pmatrix} 2+j \\ 1 \\ j \end{pmatrix} \right\rangle = 11+5j$$

$$\langle \mathbf{u}, A^* \mathbf{v} \rangle = \left\langle \begin{pmatrix} 1 \\ 1-j \\ 2 \end{pmatrix}, \begin{pmatrix} 2j \\ 3+4j \\ 6-2j \end{pmatrix} \right\rangle = 11+5j$$





Real skew-adjoint operators

- The adjoint of a real matrix is the transpose of that matrix
 - Given a square matrix A , then the entries of the transpose are:

$$\left(A^T\right)_{i,j} = a_{j,i}$$

- For $A^T = -A$, we require

$$a_{i,j} = -a_{j,i}$$

- In other words, a real square matrix is skew-adjoint if the entry on or above the diagonal equals the negation of the entry in its reflection through the diagonal
 - Because of this property, we call such matrices *skew-symmetric*





Real self-adjoint operators

Theorem

The diagonal entries of a skew-symmetric matrix are zero.

Proof:

If $a_{i,j} = -a_{j,i}$ then if $i = j$, it follows that

$$a_{i,i} = -a_{i,i}$$

This is only true if the diagonal entries are all zero. ■





An example of a symmetric matrix

- The following matrix is skew-symmetric, and is therefore is skew-adjoint:

$$A = \begin{pmatrix} 0 & -0.3 & 0.7 & 0.2 \\ 0.3 & 0 & -1.8 & 0.9 \\ -0.7 & 1.8 & 0 & 6.8 \\ -0.2 & -0.9 & -6.8 & 0 \end{pmatrix}$$





Example

- Let's check if that's actually true:

$$A = -A^T = \begin{pmatrix} 0 & 1 & -1 & -2 \\ -1 & 0 & 2 & 1 \\ 1 & -2 & 0 & -3 \\ 2 & -1 & 3 & 0 \end{pmatrix} \quad \mathbf{u} = \begin{pmatrix} -1 \\ 3 \\ 1 \\ -2 \end{pmatrix} \quad \mathbf{v} = \begin{pmatrix} 2 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

$$\langle A\mathbf{u}, \mathbf{v} \rangle = \left\langle \begin{pmatrix} 6 \\ 1 \\ -1 \\ -2 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right\rangle = 11 \quad \langle \mathbf{u}, A\mathbf{v} \rangle = \left\langle \begin{pmatrix} -1 \\ 3 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} -1 \\ -3 \\ -1 \\ 1 \end{pmatrix} \right\rangle = -11$$





Example

- If we change even one value, it no longer holds:

$$-A^T \neq A = \begin{pmatrix} 0 & 1 & -1 & -2 \\ -1 & 0 & 2 & 1 \\ -1 & -2 & 0 & -3 \\ 2 & -1 & 3 & 0 \end{pmatrix} \quad \mathbf{u} = \begin{pmatrix} -1 \\ 3 \\ 1 \\ -2 \end{pmatrix} \quad \mathbf{v} = \begin{pmatrix} 2 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

$$\langle A\mathbf{u}, \mathbf{v} \rangle = \left\langle \begin{pmatrix} 6 \\ 1 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right\rangle = 9 \neq -\langle \mathbf{u}, A\mathbf{v} \rangle = -\left\langle \begin{pmatrix} -1 \\ 3 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} -1 \\ -3 \\ -5 \\ 1 \end{pmatrix} \right\rangle = 15$$





Complex skew-adjoint operators

- The adjoint of a complex matrix is the conjugate transpose of that matrix
 - Given a square matrix A , then the entries of the conjugate transpose are:

$$\left(A^*\right)_{i,j} = a_{j,i}^*$$

- For $A = -A^*$, we require, we require

$$a_{i,j} = -a_{j,i}^*$$

- In other words, a complex square matrix is skew-adjoint if the entry on or above the diagonal equals the negation of the conjugate of the entry in its reflection through the diagonal
- Because of this property, we call such matrices *conjugate skew-symmetric*





Complex self-adjoint operators

Theorem

The diagonal entries of a conjugate skew-symmetric matrix are imaginary.

Proof:

If $a_{i,j} = -a_{j,i}^*$, then if $i = j$, it follows that

$$a_{i,i} = -a_{i,i}^*$$

This is only true if the diagonal entries are all imaginary. ■





Example

- Let's check if that's actually true:

$$A = -A^* = \begin{pmatrix} 0 & 1-j & 2+j \\ -1-j & j & 1 \\ -2+j & -1 & -2j \end{pmatrix} \quad \mathbf{u} = \begin{pmatrix} 1 \\ 1-j \\ 2 \end{pmatrix} \quad \mathbf{v} = \begin{pmatrix} 2+j \\ 1 \\ j \end{pmatrix}$$

$$\langle A\mathbf{u}, \mathbf{v} \rangle = \left\langle \begin{pmatrix} 4 \\ 2 \\ -3-2j \end{pmatrix}, \begin{pmatrix} 2+j \\ 1 \\ j \end{pmatrix} \right\rangle = 8+j$$

$$\langle \mathbf{u}, A\mathbf{v} \rangle = \left\langle \begin{pmatrix} 1 \\ 1-j \\ 2 \end{pmatrix}, \begin{pmatrix} j \\ -1-j \\ -4 \end{pmatrix} \right\rangle = -8-j$$





Example

- Note that this equality is false if the matrix is skew-symmetric:

$$-A^* \neq A = \begin{pmatrix} 0 & 1-j & 2+j \\ -1+j & 0 & 1 \\ -2-j & -1 & 0 \end{pmatrix} \quad \mathbf{u} = \begin{pmatrix} 1 \\ 1-j \\ 2 \end{pmatrix} \quad \mathbf{v} = \begin{pmatrix} 2+j \\ 1 \\ j \end{pmatrix}$$

$$\langle A\mathbf{u}, \mathbf{v} \rangle = \left\langle \begin{pmatrix} 4 \\ 1+j \\ -3 \end{pmatrix}, \begin{pmatrix} 2+j \\ 1 \\ j \end{pmatrix} \right\rangle = 9$$

$$\langle \mathbf{u}, A\mathbf{v} \rangle = \left\langle \begin{pmatrix} 1 \\ 1-j \\ 2 \end{pmatrix}, \begin{pmatrix} j \\ -3+2j \\ -4-4j \end{pmatrix} \right\rangle = -13-8j$$





Comment on terminology

- Just like the conjugate transpose is sometimes called the Hermitian transpose
 - A conjugate symmetric matrix is sometimes said to be a Hermitian matrix
 - A conjugate skew-symmetric matrix is sometimes said to be a skew-Hermitian matrix





From calculus

- Recall that a function is *even* if $f(-t) = f(t)$ for all t
 - The term *even* comes from the fact that all even powers of t share this property
 - Consider the polynomial $3t^6 + 2t^2 + 5$:
- Recall that a function is *odd* if $f(-t) = -f(t)$ for all t
 - The term *odd* comes from the fact that all odd powers of t share this property
 - Consider the polynomial $3t^7 + 2t^3 + 5t$:

$$3(-t)^6 + 2(-t)^2 + 5 = 3t^6 + 2t^2 + 5$$

$$3(-t)^7 + 2(-t)^3 + 5(-t) = -(3t^7 + 2t^3 + 5t)$$





From calculus

Theorem

Every function is the sum of an even and an odd function.

Proof:

This is a constructive proof.

$$\text{Let } f_{\text{even}}(t) \stackrel{\text{def}}{=} \frac{f(t) + f(-t)}{2} \text{ and } f_{\text{odd}}(t) \stackrel{\text{def}}{=} \frac{f(t) - f(-t)}{2}.$$

Note that $f(t) = f_{\text{even}}(t) + f_{\text{odd}}(t)$, and

$$f_{\text{even}}(-t) = \frac{f(-t) + f(-(-t))}{2} = \frac{f(-t) + f(t)}{2} = f_{\text{even}}(t)$$

$$f_{\text{odd}}(-t) = \frac{f(-t) - f(-(-t))}{2} = \frac{f(-t) - f(t)}{2} = -\frac{f(t) - f(-t)}{2} = -f_{\text{odd}}(t)$$

Thus, f is the sum of an even and an odd function. ■





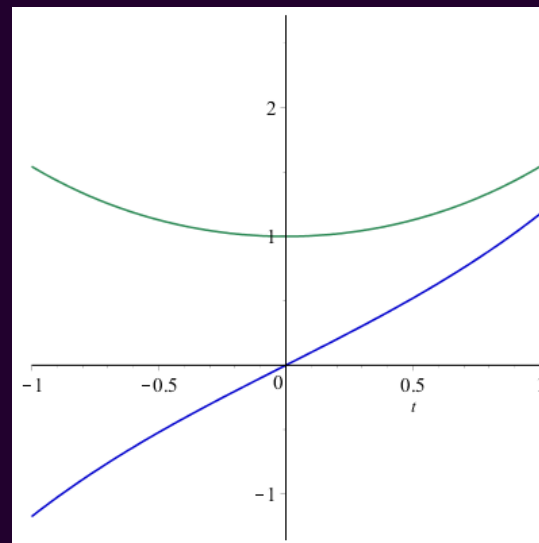
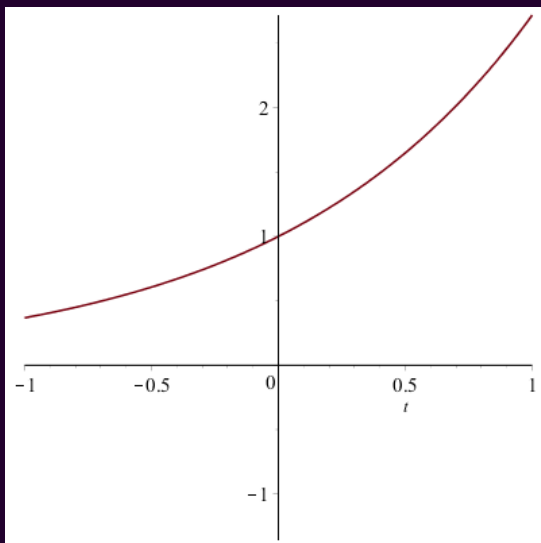
From calculus

- Note that the Taylor series of
 - An even function $f(t)$ is comprised entirely of even powers of t
 - An odd function $f(t)$ is comprised entirely of odd powers of t

$$e^t = 1 + t + \frac{t^2}{2} + \frac{t^3}{6} + \frac{t^4}{24} + \frac{t^5}{120} + \dots$$

$$e^t_{\text{even}} = 1 + \frac{t^2}{2} + \frac{t^4}{24} + \dots$$

$$e^t_{\text{odd}} = t + \frac{t^3}{6} + \frac{t^5}{120} + \dots$$





Linear operators

Theorem

Every linear operator is the sum of a self-adjoint and a skew-adjoint operator.

Proof:

Define $A_{\text{self}} \stackrel{\text{def}}{=} \frac{A + A^*}{2}$ and $A_{\text{skew}} \stackrel{\text{def}}{=} \frac{A - A^*}{2}$.

Note that

$$(A_{\text{self}})^* \stackrel{\text{def}}{=} \left(\frac{A + A^*}{2} \right)^* = \frac{A^* + (A^*)^*}{2} = \frac{A^* + A}{2} = A_{\text{self}}$$

$$(A_{\text{skew}})^* \stackrel{\text{def}}{=} \left(\frac{A - A^*}{2} \right)^* = \frac{A^* - (A^*)^*}{2} = \frac{A^* - A}{2} = -\frac{A - A^*}{2} = -A_{\text{skew}}$$

Thus, the statement is true. ■





Example

- Consider the following real matrix:

$$A = \begin{pmatrix} 3.2 & 4.5 & 2.3 & -6.2 \\ -0.3 & 4.5 & 3.1 & 1.9 \\ 0.7 & -1.8 & 2.4 & -3.5 \\ 0.2 & 0.9 & 6.8 & -0.4 \end{pmatrix}$$

$$A_{\text{self}} \stackrel{\text{def}}{=} \frac{A + A^T}{2}$$

$$= \begin{pmatrix} 3.2 & 2.1 & 1.5 & -6 \\ 2.1 & 4.5 & 0.65 & 1.4 \\ 1.5 & 0.65 & 2.4 & 1.65 \\ -3 & 1.4 & 1.65 & -0.4 \end{pmatrix}$$

$$A_{\text{skew}} \stackrel{\text{def}}{=} \frac{A - A^T}{2}$$

$$= \begin{pmatrix} 0 & 2.4 & 0.8 & -3.2 \\ -2.4 & 0 & 2.45 & 0.5 \\ -0.8 & -2.45 & 0 & -5.15 \\ 3.2 & -0.5 & 5.15 & 0 \end{pmatrix}$$





Summary

- Following this topic, you now
 - Know what a self-adjoint and skew-adjoint linear operator is
 - Understand that a real symmetric matrix is self-adjoint and a real skew-symmetric matrix is skew-adjoint
 - Understand that a complex conjugate symmetric matrix is self-adjoint, and a conjugate skew-symmetric matrix is skew-adjoint
 - Know that every linear operator is the sum of a self-adjoint and a skew-adjoint linear operator
 - Know the defining feature of a self-adjoint operator is

$$\langle A\mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{u}, A\mathbf{v} \rangle$$

and the defining feature of a skew-adjoint operator is

$$\langle A\mathbf{u}, \mathbf{v} \rangle = -\langle \mathbf{u}, A\mathbf{v} \rangle$$





References

- [1] <https://en.wikipedia.org/wiki/Self-adjoint>
- [2] https://en.wikipedia.org/wiki/Symmetric_matrix
- [3] https://en.wikipedia.org/wiki/Hermitian_matrix





Acknowledgments

None so far.





Colophon

These slides were prepared using the Cambria typeface. Mathematical equations use Times New Roman, and source code is presented using Consolas. Mathematical equations are prepared in MathType by Design Science, Inc. Examples may be formulated and checked using Maple by Maplesoft, Inc.

The photographs of flowers and a monarch butter appearing on the title slide and accenting the top of each other slide were taken at the Royal Botanical Gardens in October of 2017 by Douglas Wilhelm Harder. Please see

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